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An exact propagator for a time-dependent harmonic oscillator with a time-dependent inverse square potential

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Abstract. Using Feynman's polygonal paths for path integrals, the exact evaluation of the propagator for a time-dependent harmonic oscillator with a time-dependent inverse square potential becomes possible. The propagator at and beyond caustics is then evaluated by including the Maslov correction factor. Finally, we obtain the wavefunctions from the propagator obtained.

In this paper we consider a harmonic oscillator with time-dependent mass $M(t)$ and frequency $\omega(t)$ moving in one dimension under a time-dependent inverse square potential. The Lagrangian is given by

$$L(\dot{x}, x, t) = \tfrac{1}{2}M(t)(\dot{x}^2 - \omega^2(t)x^2) - g(t)/x^2 \tag{1}$$

where $M(t)g(t) > -\hbar^2/8$ to avoid 'the fall to the centre' (Landau and Lifshitz 1958). We shall restrict ourselves to studying the region $x \geq 0$ only since the solution in this region can be extended to $x<0$ by using analytic continuation. For later convenience, we set $r' = r(t')$, $r'' = r(t'')$ and $r_j = r(t'+j\varepsilon)$ $(\varepsilon = (t''-t')/N)$ for any function $r(t)$ of time t.

Using Feynman's polygonal paths (Feynman and Hibbs 1965), the short-time action can be written as

$$\begin{aligned} S_j(x_{j-1}, x_j) &= \frac{M_j}{2\varepsilon}(x_j - x_{j-1})^2 - \frac{\varepsilon}{2}M_j\omega_j^2x_j^2 - \frac{\varepsilon g_j}{x_jx_{j-1}} \\ &= \frac{M_j}{2}\left(\frac{x_j^2 + x_{j-1}^2}{\varepsilon} - \varepsilon\omega_j^2x_j^2\right) - \left(\frac{M_jx_jx_{j-1}}{\varepsilon} + \frac{\hbar^2(\nu_j^2 - \frac{1}{4})}{2M_jx_jx_{j-1}}\right) \end{aligned} \tag{2}$$

with $\nu_j = \frac{1}{2}(1 + 8M_jg_j/\hbar^2)^{1/2}$. Applying the asymptotic form of the modified Bessel function

$$I_{\nu_j}(u/\varepsilon) = (\varepsilon/2\pi u)^{1/2} \exp\left(\frac{u}{\varepsilon} - \frac{\varepsilon}{2u}(\nu_j^2 - \tfrac{1}{4}) + O(\varepsilon^2)\right) \tag{3}$$

for small ε, we have

$$\begin{aligned} \exp\left(\frac{i}{\hbar}\sum_{j=1}^{N} S_j(x_{j-1}, x_j)\right) &= \prod_{j=1}^{N}\left(\frac{2\pi M_jx_jx_{j-1}}{i\hbar\varepsilon}\right)^{1/2} I_{\nu_j}\left(\frac{M_jx_jx_{j-1}}{i\hbar\varepsilon}\right) \\ &\quad \times \exp\left[\frac{iM_j}{2\hbar}\left(\frac{x_j^2 + x_{j-1}^2}{\varepsilon} - \varepsilon\omega_j^2x_j^2\right)\right]. \end{aligned} \tag{4}$$

Therefore the propagator is of the form

$$K(x'', t''; x', t') = (x'x'')^{1/2} \lim_{N\to\infty} (-\mathrm{i}\beta'') \exp[\tfrac{1}{2}\mathrm{i}(\beta_1 x'^2 + \beta'' x''^2)]$$
$$\times \int_0^\infty \cdots \int_0^\infty \exp\left[\mathrm{i}\left(\sum_{j=1}^{N-1} \alpha_j x_j^2\right)\right]$$
$$\times I_{\nu''}(-\mathrm{i}\beta'' x_{N-1} x'') \prod_{j=1}^{N-1} I_{\nu_j}(-\mathrm{i}\beta_j x_{j-1} x_j)(-\mathrm{i}\beta_j) x_j \,\mathrm{d}x_j \tag{5}$$

with $\beta_j = M_j/\hbar\varepsilon$ and $\alpha_j = \beta_j(1-\omega_j^2\varepsilon^2/2)$.

Unfortunately, equation (5) cannot be evaluated exactly at present. Here we consider only the case of $M(t)g(t) = K$ (constant), or $\nu = \nu_j = \frac{1}{2}(1+8K/\hbar^2)^{1/2}$. By repeatedly using the well known integral (Peak and Inomata 1969)

$$\int_0^\infty \exp(\mathrm{i}rx^2) I_\nu(-\mathrm{i}px) I_\nu(-\mathrm{i}qx) x \,\mathrm{d}x = (\mathrm{i}/2r) \exp\left(-\frac{\mathrm{i}(p^2+q^2)}{4r}\right) I_\nu\left(\frac{-\mathrm{i}pq}{2}\right) \tag{6}$$

for $\mathrm{Re}(r) > 0$ and $\mathrm{Re}(\nu) > -1$, we obtain from equation (5) the propagator of the form

$$K(x'', t''; x', t') = -\mathrm{i}(x'x'')^{1/2} \lim_{N\to\infty} R_N \exp[\mathrm{i}(P_N x'^2 + Q_N x''^2)] I_\nu(-\mathrm{i}R_N x' x'') \tag{7}$$

where

$$R_N = \beta_1 \prod_{j=1}^{N-1} (\beta_j/2\gamma_j) \tag{8}$$

$$P_N = (\beta_1/2) - \sum_{j=1}^{N-1} (B_j^2/4\gamma_j) \tag{9}$$

$$Q_N = (\beta_N/2) - \beta_N^2/4\gamma_{N-1} \tag{10}$$

$$\gamma_j = \alpha_j - \beta_j^2/4\gamma_{j-1} \qquad (\gamma_1 = \alpha_1) \tag{11}$$

and

$$B_j = \beta_j \prod_{k=1}^{j-1} (\beta_k/2\gamma_k) \qquad (B_1 = \beta_1). \tag{12}$$

By defining $\bar{a}_j = 2\gamma_j/\beta_j$, we obtain from (11) the following recurrence relation:

$$\bar{a}_j = 2(1-\omega_j^2\varepsilon^2/2) - 1/\bar{a}_{j-1}. \tag{13}$$

Now by assuming that $\bar{a}_j = a_{j+1}/a_j$, we have

$$(a_{j+1} - 2a_j + a_{j-1})/\varepsilon^2 = -\omega_j^2 a_j - (M_j/M_{j-1} - 1)a_{j-1}/\varepsilon. \tag{14}$$

In the limit of $\varepsilon \to 0$, the above recurrence relation reduces to the following differential equation:

$$\ddot{a} + [\dot{M}(t)/M(t)]\dot{a} + \omega^2(t)a = 0 \qquad a' = 0, \dot{a}' = 1. \tag{15}$$

We can also rewrite equations (8)-(11) in terms of a_j as

$$R_N = \beta'' \prod_{j=1}^{N-1} (1/a_j) - \beta'' a_1/a_N \tag{16}$$

$$P_N = (\beta_1/2) - (a_1^2/2) \sum_{j=1}^{N-1} (\beta_j/a_j a_{j+1}) \tag{17}$$

and

$$Q_N=(\beta_N/2)(1-M''a_{N-1}/M_{N-1}a_N). \tag{18}$$

Taking $\varepsilon\to 0$, we obtain the following limit values:

$$\lim_{\varepsilon\to 0} R_N=M'b'/\hbar a'' \tag{19}$$

$$\lim_{\varepsilon\to 0} Q_N=M''\dot{a}''/2\hbar a'' \tag{20}$$

and

$$\lim_{\varepsilon\to 0} P_N=(M'/2\hbar)\lim_{\varepsilon\to 0}\left(1-(\varepsilon/M')\int_{t'}^{t''}[M(t)/a^2]\,\mathrm{d}t\right)=M'b''/2\hbar a'' \tag{21}$$

where $b=(a/\varepsilon)\{1-(\varepsilon/M')\int_{t'}^{t}[M(t)/a^2]\,\mathrm{d}t\}$ satisfies

$$\ddot{b}+[\dot{M}(t)/M(t)]b+\omega^2(t)b=0 \qquad b'=1,\ \dot{b}'=0. \tag{22}$$

Substituting (19)-(21) into (7), we obtain the propagator

$$K(x'',t'';x',t')=[-\mathrm{i}M'(x'x'')^{1/2}/\hbar a'']\times\exp[(\mathrm{i}/2\hbar a'')(M'b''x'^2+M''\dot{a}''x''^2)]I_\nu(-\mathrm{i}M'x'x''/\hbar a''). \tag{23}$$

It can easily be shown that

$$a=(\eta' s/\eta)\sin(\mu-\mu') \tag{24}$$

and

$$b=(\eta' s/\eta)[\cos(\mu-\mu')+(\dot{\eta}'/\eta')\sin(\mu-\mu')] \tag{25}$$

where $\eta=[M(t)]^{1/2}$, s and μ satisfy

$$\ddot{s}+[\omega^2(t)-\ddot{\eta}/\eta]s=1/s^3 \qquad s'=1,\ \dot{s}'=0 \tag{26}$$

and

$$s^2\dot{\mu}=1 \qquad \dot{\mu}'=1. \tag{27}$$

With the help of (24)-(27), the propagator (23) becomes

$$\begin{aligned}K(x'',t'',x';t')=&[(M'M''\dot{\mu}'\dot{\mu}''x'x'')^{1/2}/\mathrm{i}\hbar\sin\phi]\exp\{(\mathrm{i}/2\hbar)[(M'\dot{\mu}'x'^2\\&+M''\dot{\mu}''x''^2)\cot\phi+(\dot{M}'x'^2-\dot{M}''x''^2)/2+M''\dot{s}''x''^2/s'']\}\\&\times I_\nu[(M'M''\dot{\mu}'\dot{\mu}'')^{1/2}x'x''/\mathrm{i}\hbar\sin\phi] \qquad (0<\phi<\pi)\end{aligned} \tag{28}$$

with $\phi=\mu''-\mu'$.

Now, by considering (i) the number of zeros of $a(t)$ in $[t',t'']$ (Truman 1978), (ii) that the wavefunction is continuous at the points $(n+1/2)\pi$, and (iii) $I_\nu(-\mathrm{i}\zeta)=\exp(-\mathrm{i}\nu\pi)I_\nu(\mathrm{i}\zeta)$ for $\zeta>0$ (Rezende 1984), we have the propagator beyond caustics:

$$\begin{aligned}K(x'',t'';\phi\neq n\pi)=&M_\phi[(M'M''\dot{\mu}'\dot{\mu}''x'x'')^{1/2}/\mathrm{i}\hbar|\sin\phi|]\exp\{(\mathrm{i}/2\hbar)[(M'\dot{\mu}'x'^2\\&+M''\dot{\mu}''x''^2)\cot\phi+(\dot{M}'x'^2-\dot{M}''x''^2)/2+M''\dot{s}''x''^2/s'']\}\\&\times I_\nu[(M'M''\dot{\mu}'\dot{\mu}'')^{1/2}x'x''/\mathrm{i}\hbar|\sin\phi|]\end{aligned} \tag{29}$$

for $(n-1)\pi<\phi<n\pi$ (n being zero or any integer), where

$$M_\phi=\exp[-\mathrm{i}\pi(\nu+1)\,\mathrm{ent}(\phi/\pi)] \tag{30}$$

is the Maslov correction factor and $\mathrm{ent}(\phi/\pi)$ stands for the greatest integer which is less than or equal to ϕ/π. Taking $\sin\phi \to 0$, we finally obtain the propagator at caustics from (29):

$$K(x'', x'; \phi = n\pi) = (M'M''\dot\mu'\dot\mu'')^{1/2} \exp[-\mathrm{i}n\pi(\nu+1)] \times \exp\{(\mathrm{i}/2\hbar)[(\dot M'x'^2 - \dot M''x''^2)/2 + M''\dot s''x''^2/s'']\}\delta[(M'\dot\mu')^{1/2}x' - (M''\dot\mu'')^{1/2}x''] \tag{31}$$

which can easily be reduced to $\delta(x'-x'')$ when $\phi \to 0$, as it should be.

For the case of $\dot M(t) = 0$ or $M(t) = m$, (28) becomes

$$K(x'', t''; x', t') = \frac{m(\dot\mu'\dot\mu''x'x'')^{1/2}}{\mathrm{i}\hbar \sin\phi} \exp\left[\frac{\mathrm{i}m}{2\hbar}\left((\dot\mu'x'^2 + \dot\mu''x''^2)\cot\phi + \frac{\dot s''}{s''}x''^2\right)\right] \tag{32}$$

which is in agreement with (39) of Khandekar and Lawande (1975) ($\dot s' \neq 0$ in their case) and the result in the appendix of Goovaerts (1975). (29) and (31) are, respectively, equivalent to (4.17) and (4.18) of Rezende (1984) as we expect. For $\nu = \frac{1}{2}$ (without an inverse square potential), we can also return to our previous results (Cheng 1985).

Finally, by using the Hille-Hardy formula (Erdelyi 1953)

$$\frac{1}{1-u}\exp\left(-\frac{(y+z)u}{1-u}\right)I_\nu\left(\frac{2\sqrt{yzu}}{1-u}\right) = \sum_{n=0}^{\infty}\frac{n!\,u^n}{(n+\nu+1)}(yzu)^{\nu/2}L_n^\nu(y)L_n^\nu(z) \qquad (|z|<1) \tag{33}$$

with $u = \exp(-2\mathrm{i}\phi)$, $y = M''\dot\mu''x''^2/\hbar$ and $z = M'\dot\mu'x'^2/\hbar$, we obtain the wavefunctions from (29):

$$\psi_n(x, t) = \left(\frac{2(n!)}{(n+\nu+1)}\right)^{1/2} x^{\nu+1/2}\left(\frac{M(t)\dot\mu}{\hbar}\right)^{(\nu+1)/2} \exp[-\mathrm{i}(2n+\nu+1)\mu] \times \exp[(\mathrm{i}x^2/4\hbar)(2M\dot\mu + \dot M + 2M\dot s/s)] \times \exp\{-\mathrm{i}\pi(\nu+1)\,\mathrm{ent}[(\mu-\mu')/\pi]\}L_n^\nu\left(\frac{M(t)\dot\mu}{\hbar}x^2\right) \tag{34}$$

where $L_n^\nu(\ )$ are the associated Laguerre functions and $\mathrm{ent}[(\mu-\mu')/\pi]$ represents the greatest integer which is less than or equal to $(\mu-\mu')/\pi$.

As a final remark we should mention that, by using the linear space transformation (Cheng 1985)

$$y = (\xi'/\xi)x \tag{35}$$

with $\xi = [g(t)]^{1/2}$, the Lagrangian (1) becomes

$$L(\dot y, y, t) = \frac{M(t)g(t)}{2g'}\{\dot y^2 + 2y\dot y\dot\xi/\xi - [\omega^2(t) - \dot\xi^2/\xi^2]y^2\} - g'/y^2. \tag{36}$$

In doing this we transform the time-dependent term $g(t)/x^2$ into the time-independent term g'/y^2. However, we still need consider the case of $M(t)g(t) = K = M'g'$, in order to transform the Lagrangian (36) into the solvable form

$$L(\dot y, y, t) = \tfrac{1}{2}M'\left[\dot y^2 - \left(\omega^2(t) + \frac{\xi\ddot\xi - 2\dot\xi^2}{\xi^2}\right)y^2\right] - \frac{g'}{y^2} + \dot H(y, t) \tag{37}$$

with $H(y, t) = M'\dot\xi y^2/2\xi$. We will investigate the problem with more general space and time transformations and will report the results in due course.

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